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Effect Of Temperature Modulation On Rayleigh-Bénard Convection In Boussinesq-Stokes Suspension

Ravichandra Nayakar S. N[1], Mamatha H.S.[2]

[1]*Department of Mathematics, University BDT College of Engineering, Davangere 577 004, India*
Email: mathsravi@gmail.com

[2]*Department of Mathematics, University BDT College of Engineering, Davangere 577 004, India*
Email: hsmamathaubdt@gmail.com

ABSTRACT

This study comprehensively investigates the influence of time-periodic boundary temperature on the onset of convection in a Boussinesq-Stokes suspension, a fluid system characterized by unique viscosity and density properties. The research focuses on how both temporal and spatial variations affect the distribution of the basic state temperature within the suspension layer, which is heated from below and cooled from above. To analyze the stability and determine the eigenvalue associated with this phenomenon, a perturbation approach is employed, accounting for the magnitude of boundary temperature perturbations. This method allows for a detailed examination of the system's response to modulated thermal conditions. A corrected Rayleigh number is derived to assess the stability of the system, particularly to evaluate the potential for sub-critical instability, where convection may occur at lower Rayleigh numbers than predicted by classical theory. The findings are compared with exact solutions to validate the results, providing insights into controlling convection through temperature modulation. The study highlights the impact of modulation frequency and amplitude on convection onset, offering valuable implications for fluid dynamics applications involving suspensions.

Keywords: Rayleigh-Bénard Convection, Boussinesq-Stokes Suspension, Temperature Modulation, Critical Rayleigh Number, Thermal Stability.

I. Introduction

A simple temperature profile that depends on both place and time is necessary since one of the effective ways of preventing convection by maintaining a non-uniform temperature gradient originates from transient heating or cooling at the borders. The stability of a horizontal layer of a viscous fluid heated from below was studied by Venezian (1969) under conditions of a constant temperature gradient between the layer's surfaces and a sinusoidal disturbance to the wall temperatures that varied with time. Although time-periodic modulation of the wall temperatures stabilizes at low frequencies, it destabilizes the initiation of convection throughout a broad range of modulation frequencies, as shown later by Yih and Li (1972). According to the research, changing the frequency of the enforced temperature modulation affects the critical Rayleigh number (which corresponds to the start of convection) in certain situations, and modifying this modulation may either speed up or slow down the start of instability. Topics covered in Lage's (1993) research include oscillatory heating and the effects of vertical density gradients on convection that vary with time. Numerous researchers have examined the stability of various fluid layers when exposed to heat modulation. (see Venezian 1969, Liu 2004, Siddheshwar and Pranesh 1999, 2000, Siddheshwar and Abraham 2003, Mahabaleswar 2007).

For fluid applications involving suspensions, the convection control issue is relevant and interesting. This is why we investigate the issue of controlling convection by temperature regulation. In a Boussinesq-Stokes suspension layer that is heated from below, we find the beginning of convection when the wall temperatures are subjected to a time-periodic disturbance in addition to a fixed temperature differential between them.

II. Mathematical Formulation and Solution

Think of a Boussinesq-Stokes layer hot from below and cold from above, contained between two endless horizontal surfaces spaced *'d'* apart (refer to Fig. 1). In a Cartesian coordinate system, the z-axis extends vertically upward from its origin at the bottom border.

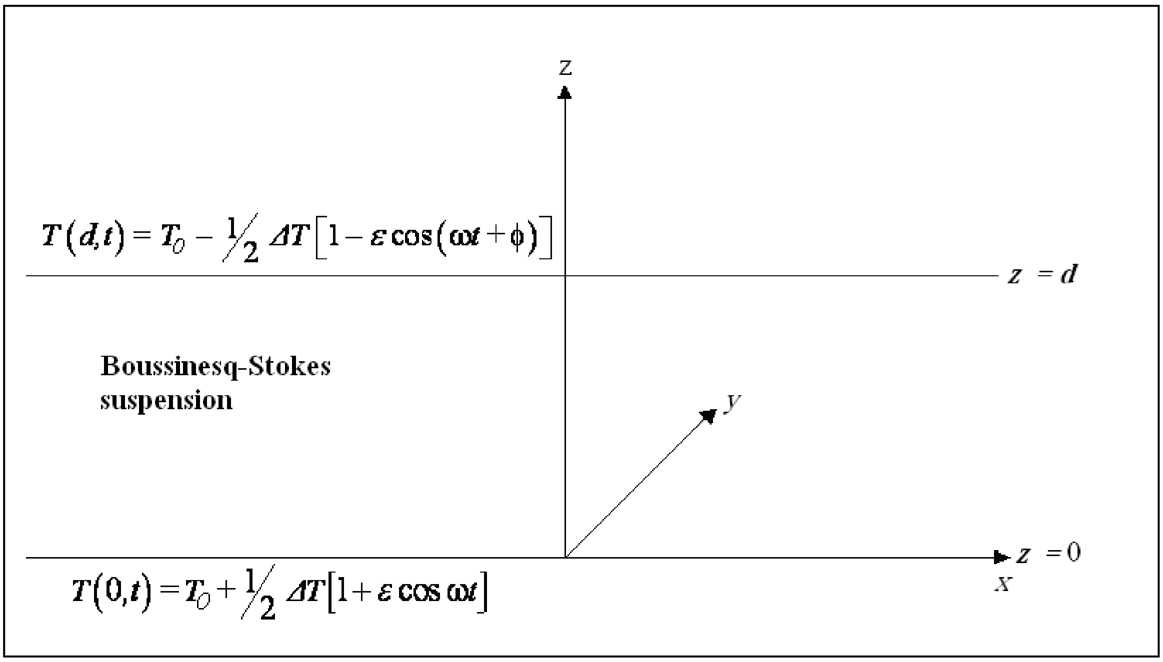

Figure 1. Physical Configuration

The surface temperatures are:

$$T(0,t) = T_0 + \frac{1}{2}\Delta T\left[1+\varepsilon\cos\omega t\right] \quad \text{at} \quad z=0 \tag{2.1}$$

and

$$T(d,t) = T_0 - \frac{1}{2}\Delta T\left[1-\varepsilon\cos(\omega t+\phi)\right] \quad \text{at} \quad z=d. \tag{2.2}$$

The governing equations are:

Continuity Equation

$$\nabla\cdot\vec{q} = 0, \tag{2.3}$$

Conservation of Linear Momentum

$$\rho_0\left[\frac{\partial\vec{q}}{\partial t}+(\vec{q}\cdot\nabla)\vec{q}\right] = -\nabla p-\rho g\hat{k}+\mu\nabla^2\vec{q}-\mu'\nabla^4\vec{q}', \tag{2.4}$$

Conservation of Energy

$$\frac{\partial T}{\partial t}+\vec{q}\cdot\nabla T = \chi\nabla^2 T, \tag{2.5}$$

Equation of State

$$\rho = \rho_0\left[1-\alpha(T-T_0)\right], \tag{2.6}$$

III. Basic State

This research aims to examine the stability of a quiescent state against tiny perturbations applied to the fundamental state. The fundamental condition of the liquid in a quiescent state is characterized by

$$\vec{q}_b = \vec{0},\ T = T_b(z,t),\ \ p = p_b(z,t),\ \ \rho = \rho_b(z,t). \qquad (3.1)$$

The temperature T_b, pressure p_b and density ρ_b satisfy

$$\frac{\partial T_b}{\partial t} = \chi \frac{\partial^2 T_b}{\partial z^2}, \qquad (3.2)$$

$$-\frac{\partial p_b}{\partial z} = \rho_b g \qquad (3.3) \text{ and}$$

$$\rho_b = \rho_o\left[1 - \alpha\left(T_b - T_0\right)\right]. \qquad (3.4)$$

Solution to equation (3.2) that fulfills thermal boundary conditions (2.1) and (2.2) is

$$T_b = T_0 + \frac{\Delta T}{2}\left(1 - \frac{2z}{d}\right) + \varepsilon \ \mathrm{Re}\left\{\left[a(\lambda)e^{\frac{\lambda z}{d}} + a(-\lambda)e^{-\frac{\lambda z}{d}}\right]e^{-i\omega t}\right\}. \qquad (3.5)$$

Here Re {….} denotes the real part of {….} and

$$\lambda = (1-i)\sqrt{\frac{\omega d^2}{2\chi}},\ a(\lambda) = \frac{\Delta T}{2}\left\{\frac{e^{-i\varphi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}\right\}. \qquad (3.6)$$

IV. Linear Stability Theory

Let the fundamental state be disrupted by an infinitesimal disruption. The physical quantities may now be considered as

$$\vec{q} = \vec{q}_b + \vec{q}',\ \ \rho = \rho_b(z) + \rho',\ p = p_b(z) + p',\ \ T = T_b + \theta. \qquad (4.1)$$

The prime denotes that the quantities are infinitesimal perturbations.

By substituting equation (4.1) into equations (2.3) through (2.6) and using the fundamental state solution, we get the linearized equations that regulate the infinitesimal perturbations in the following form.

$$\nabla \cdot \vec{q}' = 0, \qquad (4.2)$$

$$\rho_0\left[\frac{\partial \vec{q}'}{\partial t}\right] = -\nabla p' - \rho' g\,\hat{k} + \mu \nabla^2 \vec{q}' - \mu' \nabla^4 \vec{q}', \qquad (4.3)$$

$$\frac{\partial \theta}{\partial t} + W'\frac{\partial T_b}{\partial z} = \chi \nabla^2 \theta, \qquad (4.4)$$

$$\rho' = -\alpha\, \rho_0\, \theta. \qquad (4.5)$$

The perturbation equations (4.2)-(4.5) are non-dimensionalized using the following definition:

$$(x^*, y^*, z^*) = \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right),\quad t^* = \frac{\chi t}{d^2},\quad W^* = \frac{W'}{\chi/d},\quad \theta^* = \frac{\theta}{\Delta T}. \qquad (4.6)$$

By applying the curl operator twice to equation (4.3) and using equation (4.5), followed by non-dimensionalizing the resultant equation and equation (4.4) with the aid of equation (4.6), we get

$$\left[\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2 + C\nabla^4\right]\nabla^2 W = R\nabla_1^2 \theta, \qquad (4.7)$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = -W\frac{\partial T_0}{\partial z}. \qquad (4.8)$$

In equation (4.8), $\frac{\partial T_0}{\partial z}$ is the non-dimensional form of $\frac{\partial T_b}{\partial z}$ and is given by:

$$\frac{\partial T_0}{\partial z} = -1 + \varepsilon f \tag{4.9}$$

$$f = \mathrm{Re}\left\{ \left[A(\lambda)e^{\lambda z} + A(-\lambda)e^{-\lambda z} \right] e^{-i\omega t} \right\}$$

where
and

$$A(\lambda) = \frac{\lambda}{2}\left\{ \frac{e^{-i\varphi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}} \right\}. \tag{4.10}$$

Equations (4.7) - (4.8) are solved subject to the conditions

$$W = \nabla^2 W = \nabla^4 W = \theta = 0, \quad z = 0, 1. \tag{4.11}$$

Eliminating θ between from equations (4.7) - (4.8), we get an equation for W in the form

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\left[\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2 + C\nabla^4\right]W = -R\frac{\partial T_0}{\partial z}\nabla_1^2 W. \tag{4.12}$$

In order to enable analytical treatment, we just take into account free-free and isothermal circumstances. By using the vertical component of velocity W alone to represent the boundary conditions (4.11), we may comply with Eq. (4.12). When asterisks are removed, the boundary conditions are expressed as (Siddheshwar and Pranesh 2004):

$$W = \frac{\partial^2 W}{\partial z^2} = \frac{\partial^4 W}{\partial z^4} = \frac{\partial^6 W}{\partial z^6} = 0 \qquad \text{at} \quad z = 0, 1. \tag{4.13}$$

V. Perturbation Procedure

We consider thermal modulation and seek the eigenfunction W and eigenvalue R of equation (4.12) for the basic temperature distribution (4.9) that departs from the linear profile $\left(\partial T_0 / \partial z\right) = -1$ by using quantities of order ε. Thus, the eigenvalue of the present problem differs from those of the classical Rayleigh- Bénard convection by quantities of ε. We seek the solution of equation (4.12) in the form

$$(R, W) = (R_0, W_0) + \varepsilon(R_1, W_1) + \varepsilon^2(R_2, W_2) + ------. \tag{5.1}$$

Malkus and Veronis (1958) first used this type of expansion in connection with the study of finite-amplitude convection. Here W_0 and R_0 are the eigenfunction and eigenvalue respectively of the unmodulated system and $(W_i, R_i)(i \geq 1)$ are the corrections due to modulation to W_0 and R_0.

In keeping with Venezian (1969), we insert the expansion (5.1) into Eq. (4.12), and we equal the coefficients of different powers of ε on both sides of the equation. The resulting set of equations is as follows:

$$LW_0 = 0, \tag{5.2}$$

$$LW_1 = R_1\nabla_1^2 W_0 - R_0 f\nabla_1^2 W_0, \tag{5.3}$$

$$LW_2 = R_1\nabla_1^2 W_1 + R_2\nabla_1^2 W_0 - R_0 f\nabla_1^2 W_1 - R_1 f\nabla_1^2 W_0, \tag{5.4}$$

where

$$L = \left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\left[\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2 + C\nabla^4\right] - R_0\nabla_1^2. \tag{5.5}$$

Each W_n is required to satisfy the boundary conditions (4.13). Equation (5.2), obtained at $o(\varepsilon^0)$, is the one utilized to examine convection in a horizontal layer of Boussinesq-Stokes suspension subjected to a uniform gravitational field (refer to Siddheshwar and Pranesh 2004:). Solutions that are only somewhat stable include

$$W_0 = \sin \pi z, \tag{5.6}$$

with corresponding eigenvalue R_o given by (see Siddheshwar and Pranesh 2004)

$$R_0 = \frac{\left(\pi^2 + a^2\right)^3 \left(1 + C\left(\pi^2 + a^2\right)\right)}{a^2}. \tag{5.7}$$

It is well known that for $C = 0$, R_0 assumes the minimum value $R_{0c} = 27\pi^4/4$ at $a_c = \pi/\sqrt{2}$.

At the second stage, the equation for W_1 becomes

$$L\,W_1 = R_1 a^2 \sin \pi z - R_0 a^2 f \sin \pi z. \tag{5.8}$$

The solution to the inhomogeneous equation (5.8) is problematic since it includes a resonance component. For equation 5.8 to be solvable, the time-independent term on the right-hand side must be perpendicular to the null space of operator L. L (Venezian 1969). Since f varies sinusoidally in time, the only steady term is $-R_1 a^2 \sin \pi z$ so that R_1 must be zero to yield a non-trivial solution. In fact, all the odd coefficients R_1, R_3, ... are zero. Therefore, equation (5.8) becomes

$$L\,W_1 = -R_0 a^2 f \sin(\pi z). \tag{5.9}$$

Solving Eq.(5.9), subject to Eq.(4.13), yields W_1 and substituting this into (5.4), with W_0 given by Eq.(5.6) and $R_1 = 0$, the Venezian (1969) procedure yields R_2 in the form:

$$R_2 = \frac{R_0^2 a_0^2}{4} \operatorname{Re} \sum \frac{\left|B_n(\lambda)\right|^2}{\left|L(\omega,n)\right|^2} \left[L(\omega,n) + L^*(\omega,n) \right], \tag{5.10}$$

where

$$B_n(\lambda) = \frac{-2n\pi^2\lambda^2 \left[e^{\lambda} - e^{-\lambda} + (-1)^n \left(e^{-\lambda - i\varphi} - e^{\lambda - i\varphi} \right) \right]}{e^{\lambda} - e^{-\lambda} \left[\lambda^2 + (n+1)^2 \pi^2 \right] \left[\lambda^2 + (n-1)^2 \pi^2 \right]}, \tag{5.11}$$

$$\lambda = (1-i)\sqrt{\frac{\gamma}{2}}, \tag{5.12}$$

$$L(\omega,n) = \left[\begin{array}{l} \left(\frac{\omega^2}{Pr}\left(n^2\pi^2 + a_0^2\right) - \left(1 + C\left(n^2\pi^2 + a_0^2\right)\right) \right) \left(n^2\pi^2 + a_0^2\right)^3 + \\ \left(\pi^2 + a_0^2\right)^3 \left(1 + C\left(\pi^2 + a_0^2\right)\right) + i\left(\left(1 + \frac{1}{Pr}\right) + C\left(n^2\pi^2 + a_0^2\right) \right) \end{array} \right] \tag{5.13}$$

$$+ i\omega \left[\left(1 + \frac{1}{Pr}\right) \left(n^2\pi^2 + a_0^2\right) \right],$$

and $L^*(\omega,n)$ are the conjugates of $L(\omega,n)$ respectively. Three cases of thermal modulation are considered in the study:

Case A: In –Phase Modulation

When the oscillating temperature field is ***symmetric*** so that the wall temperatures are modulated in phase (with $\varphi = 0$).

In this case *n* is even or odd.

Case B: Out-Of-Phase Modulation

When the wall temperature field is ***asymmetric*** corresponding to out-of-phase modulation (with $\varphi = \pi$).

In this case *n* is odd.

Case C: Only Lower Wall Modulation

When only the temperature of the ***only lower wall*** is modulated, the upper plate being held at a constant temperature. This case corresponds to $\varphi = -i\infty$

In this case *n* takes both even and odd values.

The infinite series (5.10) converges rapidly.

VI. Minimum Rayleigh Number for Convection

The value of R obtained by this procedure is the eigenvalue corresponding to the eigenfunction W which, though oscillating, remains bounded in time. Since R is a function of the horizontal wave number a and the amplitude of perturbation ε, as noted in equation (5.1) we have

$$R(a,\varepsilon) = R_0(a) + \varepsilon^2 R_2(a) + ... \tag{6.1}$$

The smallest value of the Rayleigh number R_c occurs at $a = a_c$. This critical value of wave number a_c occurs when $\partial R/\partial a = 0$. Similarly, we expand a_c in powers of ε as:

$$a_c = a_0 + \varepsilon_1 a_1 + \varepsilon^2 a_2 + \cdots. \tag{6.2}$$

By using the Taylor expansion, the condition $\frac{\partial R}{\partial a} = 0$ at $a = a_c$ can be written as

$$\frac{\partial R_0}{\partial a_0} + \varepsilon\left(\frac{\partial^2 R_0}{\partial a_0^2}\right)a_1 + \varepsilon^2\left[\frac{1}{2}\left(\frac{\partial^3 R_0}{\partial a_0^3}\right)a_1^2 + \left(\frac{\partial^2 R_0}{\partial a_0^2}\right)a_2 + \left(\frac{\partial R_2}{\partial a_0}\right)\right] + ... = 0. \tag{6.3}$$

Equating the coefficients of the like powers of ε^2 to zero, we obtain

$$\frac{\partial R_0}{\partial a_0} = 0,\ a_1 = 0,\ a_2 = -\left(\frac{\partial R_2}{\partial a_0}\right)\Big/\left(\frac{\partial^2 R_0}{\partial a_0^2}\right). \tag{6.4}$$

A similar expansion of R_c gives

$$R_c(\varepsilon) = R_{0c} + \varepsilon^2 R_{2c} + \varepsilon^4 R_{4c} + \cdots = R_{0c}(a_0) + \varepsilon^2 R_{2c}(a_0) + \cdots \tag{6.5}$$

Now, R_c is determined up to order ε^2 by evaluating R_0 and R_2 at $a = a_0$.

It was shown by Venezian (1969) that the critical value of R, i.e., R_c, to evaluate the critical value of R_2 is determined to $o\left(\varepsilon^2\right)$ by evaluating R_0 and R_2 at $a = a_0$. To evaluate the critical value of R_2

(denoted by R_{2c}) one has to substitute $a = a_0$ in R_2 where a_0 is the value at which R_0 given by Eq. (5.7) is minimum.

VII. Results and Discussion

In this research, the consequences of temperature modulation upon convection initiation in Boussinesq-Stokes suspensions are studied analytically. The following consequences on the classical Rayleigh-Bénard issue are taken into account in accordance with the stated reasoning on convection control:

(i) inhibition of convection by suspended particles, and
(ii) temperature modulation.

Pair stress parameters and stand for these two impacts, respectively. One thing to keep in mind before diving into the findings shown in Figures 2–7 is that in Boussinesq–Stokes suspension, the oscillatory convection mode is not a factor. Additionally, we see that, in comparison to Newtonian liquid, the Prandtl number of Boussinesq-Stokes suspension is greater.

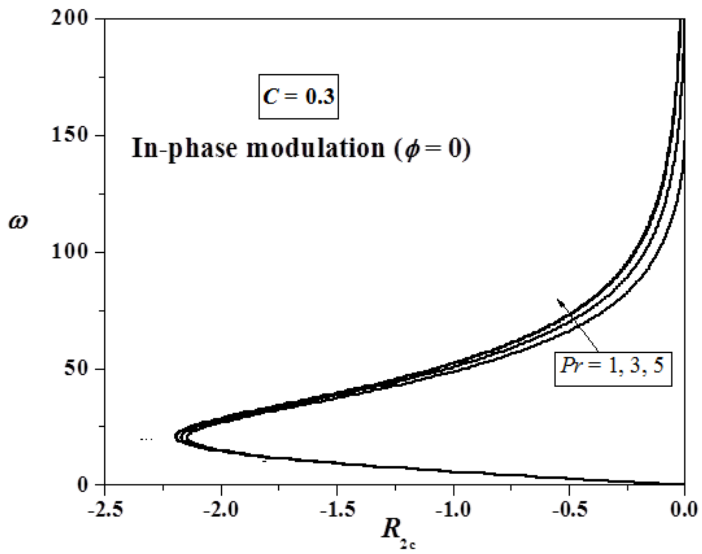

Figure 2. Plot of R_{2C} versus ω for different values of P_r (C= 0.3 & $\varphi = 0$)

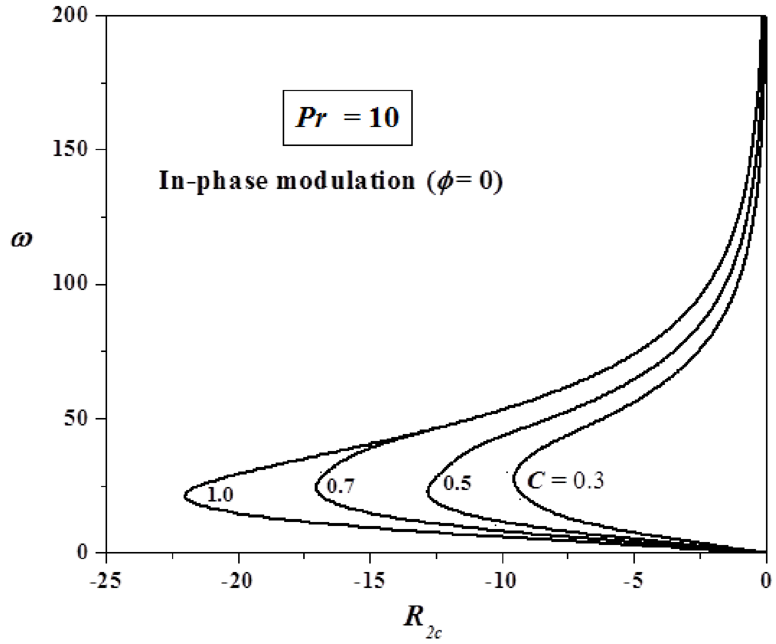

Figure 3. Plot of R_{2C} versus ω for different values of C (P_r = 10 & $\varphi = 0$)

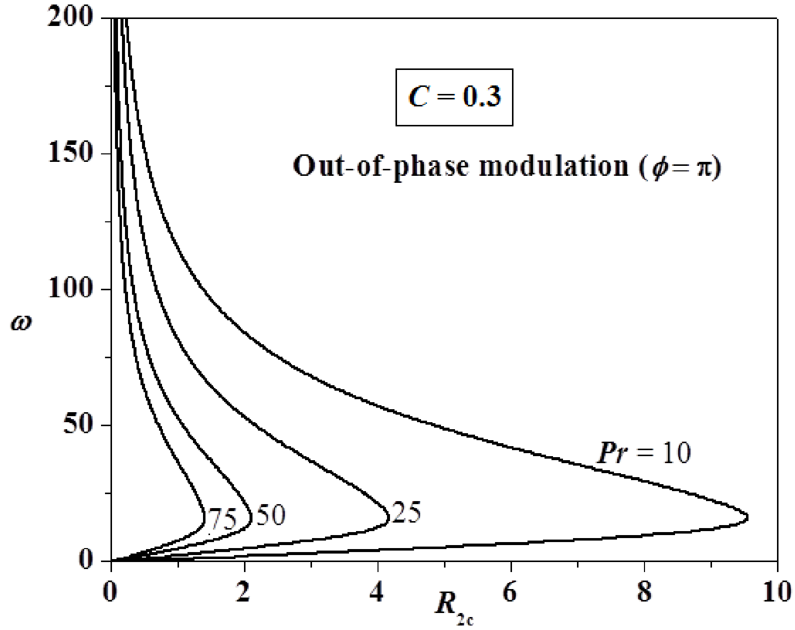

Figure 4. Plot of R_{2C} versus ω for different values of P_r (C= 0.3 & $\varphi = \pi$)

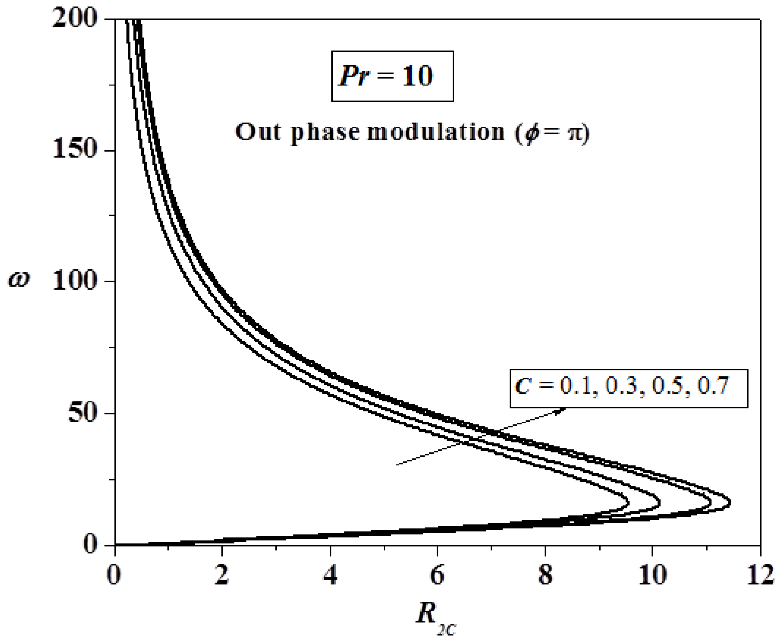

Figure 5. Plot of R_{2C} versus ω for different values of C (P_r = 10 & $\varphi = \pi$)

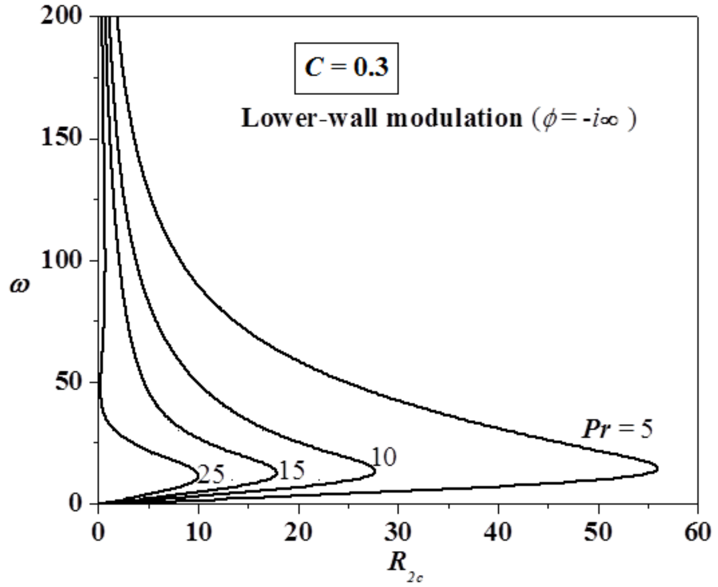

Figure 6. Plot of R_{2C} versus ω for different values of P_r (C= 0.3 & $\varphi = -i\infty$)

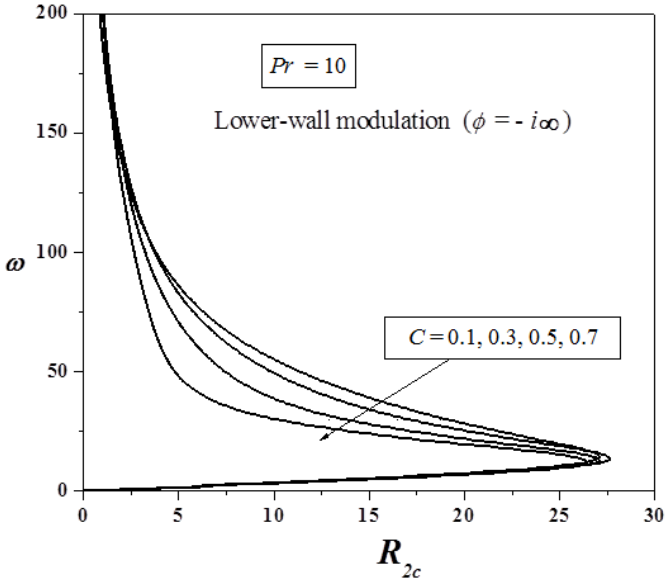

Figure 7. Plot of R_{2C} versus ω for different values of C ($P_r = 10$ & $\varphi = -i\infty$)

In the case of thermal modulation the amplitude ε is small compared with the imposed steady temperature difference. The validity of the results obtained here depends on the value of the modulating frequency ω. When $\omega \to 0$, the period of modulation is large and hence the disturbance grows to such an extent as to make finite amplitude effects important. When $\omega \to \infty$, $R_{2c} \to 0$, thus the effect of modulation becomes small. In view of this, we choose only moderate values of ω in our present study.

We now discuss the result arrived at in the paper. Three different cases are considered:

Case A: In-phase modulation,

Case B: Out-of-phase modulation and

Case C: Only lower-wall modulation.

Fig. 2 is the plot of the critical value of R_2, ie., R_{2c}, versus frequency ω for different values of Prandtl number Pr and fixed values of couple stress parameter C in respect of in case A. We observe that as Pr increases, R_{2c} becomes more and more negative. This means that Boussinesq-Stokes suspension more vulnerable than Newtonian fluids to destabilization by modulation. It is appropriate to note here that Pr does not affect the R_0 -part of R (see Eq.5.7). It affects only R_2, as R_0 is the Rayleigh number of the unmodulated system. It is also observed that in the case of Boussinesq-Stokes suspension subcritical motions are possible for in phase- modulation.

Fig. 3 is the plot of R_{2c} versus ω for different values of C and fixed values of Pr in the case of in-phase modulation. We observe from the figure that as C increases R_{2c} becomes more and more negative. It is interesting to note that for a given value of C, R_{2c} decreases with ω for small values of ω and increases with ω for moderate values of ω. Thus, small values of ω destabilize and moderate values of ω stabilize the system. Reason being, at low modulation frequencies, the whole liquid layer experiences the impact of thermal modulation on the temperature field. A temperature profile with in-phase modulated plates has both a stationary straight line segment and a time-varying parabolic profile. The parabolic portion of the profile grows in importance in relation to the modulation amplitude. It is recognized that convection occurs at lower Rayleigh numbers than those anticipated by the linear theory due to finite-amplitude instabilities in a parabolic profile.

Fig.4 is a plot of R_{2c} versus ω for different values of *Pr* and fixed value of other parameters in respect of out-of-phase modulation. We find that even though R_{2c} decreases with increase in *Pr* it does not become negative. Thus subcritical motion is ruled out in the case of out-of-phase modulation. R_{2c}

We now discuss the results pertaining to out-of-phase modulation. Comparing Figs.2 and 4 and Figs. 3 and 5 respectively we can conclude that R_{2c} is positive for out-of-phase whereas it is negative for in-phase modulation. The above results are due to the fact that in the case of out-of-phase modulation the temperature field has essentially a linear gradient varying in time, so that the instantaneous Rayleigh number is supercritical for half a cycle and subcritical during the other half cycle (see Venezian 1969).

The above results on the effect of various parameters on R_{2c} for out-of-phase modulation do not qualitatively change for the case of temperature modulation of just the lower boundary. This is illustrated with the help of Figs. 6 and 7. The physical explanation is the same as in out-of-phase modulation (see Venezian 1969).

The results of the study throw light on an external means of controlling convection in Boussinesq-Stokes suspensions, either advancing or delaying convection by thermal modulation. It is also observed that for large frequencies, the effect of modulation disappears.

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